HISTORIA MATHEMATICA 2 (1975), 439-447

THE MATHEMATICIAN, THE HISTORIAN, AND THE HISTORY OF MATHEMATICS

BY JUDITH V. GRABINER
SMALL COLLEGE, CALIFORNIA STATE COLLEGE, DOMINGUEZ HILLS

The historian's basic questions, whether he is a historian of mathematics or of political institutions, are: what was the past like? and how did the present come to be? The second question--how did the present come to be?--is the central one in the history of mathematics, whether done by historian or mathematician. But the historian's view of both past and present is quite different from that of the mathematician. The historian is interested in the past in its full richness, and sees any present fact as conditioned by a complex chain of causes in an almost unlimited past. The mathematician instead is oriented toward the present, and toward past mathematics chiefly insofar as it led to important present mathematics. [a]

I

What questions do mathematicians generally ask about the history of mathematics? "When was this concept first defined, and what problems led to its definition?" "Who first proved this theorem, and how did he do it?" "Is the proof correct by modern standards?" The mathematician begins with mathematics that is important now, and looks backwards for its antecedents. To a mathematician, all mathematics is contemporary; as Littlewood put it [A7, p. 81], the ancient Greeks were "Fellows of another College." True and significant mathematics is true and significant, whenever it may have been done.

The history of mathematics as written by mathematicians tends to be technical, to focus on the content of specific papers. It is written on a high mathematical level, and deals with significant mathematics. The title of E. T. Bell's *The Development of Mathematics* reflects the mathematician's view. The mathematician looks at the development of *mathematics,* as the result of a chronologically and logically connected series of papers; he does not look at it as the work of people living in considerably different historical settings.

II

How is the historian different? First, it goes without saying that, in asking what the past was like, the historian will be more concerned than the mathematician about the non-mathematical, as well as the mathematical past. More surprising is that even when dealing with strictly technical questions, the historian may view things differently from the mathematician.

While the mathematician sees the past as part of the present, the historian sees the present as laden with archaeological relics from the past; he sees everything in the present as having many and diverse roots in the past, and as the end of long, complex processes. Many things we take for granted are neither logical nor natural. They might even appear arbitrary, but they are, instead, the products of particular historical situations. For example, the use of the letter "epsilon," as in delta-epsilon proofs, appears arbitrary, but it in fact records the origin of the use of inequalities in proofs in analysis. The origin was in the study of approximations, the notation "epsilon" is Cauchy's, and the letter seems to stand for "error." [b] Historians love this sort of explanation, and constantly search for ones like it. To take another kind of example, isn't it amazing that the standard proof that $\sqrt{2}$ is irrational is over two thousand years old? [c]

Another essential difference between the historian and the mathematician is this: the historian of mathematics will ask himself what the *total* mathematical past in some particular time-period was like. He will steep himself in many aspects of that mathematical past, not just those which have an obvious bearing on the antecedents of the particular mathematical development whose history he is tracing. [A2;A3;A4;A8] Let us see how this feature of the historian's approach can be of value to the mathematician interested in the history of mathematics. Obviously it is easier to find the antecedents of present ideas when one knows where to look. The better one knows the past, the wider the variety of places he can investigate. In addition, through familiarity with specific types of sources, the historian knows where to find the answers to particular types of questions. By contrast, someone relatively unfamiliar with the time period in question will not always understand what past mathematicians were trying to do, and will find that the terms used then did not always mean what they mean today. [d]

Let me give some examples to clarify this general point. There is a widespread impression that eighteenth-century mathematicians were very cavalier in their treatment of convergence, and it is sometimes even said that they assumed that once they had shown that the n^{th} term of a series went to zero, the series converged. Did eighteenth-century mathematicians in fact make this error? Hadn't they ever heard of the harmonic series? My own sense of eighteenth-century mathematics says that eighteenth-century mathematicians weren't that incompetent. They knew the divergence of the harmonic series [e]. People like Euler, D'Alembert, Lagrange, and Laplace were not hopelessly confused. In fact, D'Alembert and Lagrange investigated the remainders of specific infinite series and tried to find bounds on the value of those remainders [f]. Why, then, did even these men sometimes say "the series converges" when they had shown only that the nth term goes to zero? Because, in the eighteenth century, the term "converge" was used in different ways; sometimes, it was used as we use it; often, however, it was used to say that the nth term went to zero or that the terms of the series got smaller [g]. This conclusion, which I reached after reading numerous eighteenth-century papers, can be reliably verified by looking at the Diderot-D'Alembert *Encyclopédie* [h]. The modern definition of convergent series -- that the partial sums of the series have a limit -- was established by Cauchy [B6 (2), vol. 3, p. 114]. It is hard to avoid reading this modern meaning back into eighteenth-century mathematics. But this linguistic point, once understood, makes sense out of much eighteenth-century work on infinite series.

Another advantage of knowing the mathematical past is that the historian can construct a total picture of the background of some specific modern achievement. Rather than just looking at the major papers on the same topic, he may find the antecedents of some modern theories in unlikely places. A well-known example is the way the general definition of function came, not merely out of attempts to describe the class of all known algebraic expressions, but, more importantly, from the attempts to characterize the solutions to the partial differential equation for the vibrating string [i]. Another example may be found in the way Thomas Hawkins in his *Lebesgue's theory of Integration* [E14] has presented the full nineteenth-century background, drawing on a wide variety of mathematical work besides earlier ideas on integration.

For another example, consider Cauchy's definition of the derivative and the proofs of theorems based on that definition. In looking at the introductory sections of eighteenth-century calculus books, one finds a long string of definitions of derivatives, and debates about their nature -- debates stemming from the attack on the foundations of the calculus by Bishop Berkeley. One might well view each of these old definitions and polemics as major contributors to Cauchy's final formulation. However, what was more important than the explicit verbal definition Cauchy gave for the derivative are the associated inequality proof-techniques he pioneered. And these techniques came from elsewhere. The basic inequality property Cauchy used to define the derivative came to him from Lagrange's work on the Lagrange remainder in the latter's lectures on the calculus at the *Ecole polytechnique* [j]. The inequality proof-techniques themselves were developed largely in the study of algebraic approximations in the eighteenth century [k].

Let us now take up a characteristic of the historian which we have not yet considered. We expect the historian to know the general history of a particular time as well as the mathematics of that time. He should have a sense of what it was like to be a person, not just a mathematician, at that time. Sometimes such knowledge has great explanatory value. One would not want to treat the history of the foundations of the calculus without knowing about the attacks on the calculus by the theologian Berkeley [B3, ch. VI]; or the flowering of seventeenth-and eighteenth-century mathematics without reference to the contemporary explosion in the natural sciences, especially Newtonian physics [A6]. One cannot treat the growth of the French school of mathematics in the nineteenth century without mentioning a major cause -- the founding by the French revolutionary government of the *Ecole polytechnique,* providing employment and a first-rate mathematical community for its faculty, and an excellent mathematical education for its students [ℓ]. One would not want to explain the relative absence of women in the ranks of nineteenth-century mathematicians without referring to the lack of access to higher education for women in Europe at a time when mathematics was so specialized and advanced that formal training was essential [m].

By virtue of his training, the historian has been exposed to a number of general historiographical questions and is used to hearing them asked. For instance, there are theories of the nature of scientific change, like Thomas Kuhn's theory of scientific revolutions [A9]. Again, there are sociologically based theories like Robert Merton's analysis of priority controversies in science [A13]. The historian of mathematics, without having to become a disciple of Kuhn or Merton, can use such theories to help ask fruitful questions about the past of mathematics, and about the time period in which the mathematics occurred. He has the questions already at hand, and need not figure them out from first principles.

III

Our description of the possible contributions of historians and mathematicians to the writing of the history of mathematics has required, as well, some description of what the history of mathematics is like. Let us know turn to a different, but related question. What value has the history of mathematics, whether done by historian or mathematician? Of course, there is an inherent fascination in any history, and work in the history of mathematics certainly should be an element in the history of human culture in general. But another essential use exists -- for the mathematician -- in teaching and understanding mathematics.

Historical background can help teach mathematics in three ways. First, the history can help the teacher understand the inherent difficulty of certain concepts. A concept which took hundreds of years to develop is probably hard, and the historical difficulties may well resemble student difficulties.

Second, understanding how a mathematical idea arose can help motivate students. It helps answer questions like, "Why might somebody want to think about it in this particular way?" [n]. Isolated historical comments, of course, do not make history. What one would like to do for one's students -- and for that matter, for oneself -- is to give them a sense of how the whole subject developed, and how the whole background of the subject fits together. Such a sense would motivate not just one concept or proof, but the entire subject.

Third, the historical background can help the student -- or the mathematician -- see how mathematics fits in with the rest of human thought; how Descartes the mathematician relates to Descartes the philosopher; how the rise of German mathematics in the mid-nineteenth century fits into the rise of German science, technology, and national power at that time. To see past mathematics in its historical context helps to see present mathematics in its philosophical, scientific, and social context, and to have a better understanding of the place of mathematics in the world.

IV

If the history of mathematics is indeed to be used in these ways, we need more of it. There now exists a technical literature which has established what the important results and their major antecedents are in many areas [o]. There is a need now for more studies on the full historical background of many subjects in modern mathematics -- the theory of functions of a complex variable, for instance, or the rise of abstract algebra, or the philosophical and mathematical impact of non-Euclidean geometry. And the existing work needs to be made more available through the offering of full-scale courses in the history of mathematics, placing the monographs of Boyer and Hawkins along with the general histories of mathematics on the library shelves, and ordering *Archive for the History of the Exact Sciences* and *Historia Mathematica* for the departmental library along with the *Bulletin* and the *Monthly*. Finally, there are now too few historians of mathematics. The path for the historian of mathematics is difficult; he needs the historian's training, but also needs to know a great deal of mathematics. The history of science is itself a young and relatively small profession; the number of historians of mathematics, because of the types of knowledge needed, is even smaller. Still the need for such people is apparent.

Even if historians of mathematics were legion, however, the contribution of mathematicians to the history of mathematics would remain crucial. Mathematicians, of course, bring a higher level of mathematical knowledge to any historical task. Historians of mathematics should certainly know the mathematics whose history they are writing. But mathematicians are still needed -- and not just because they know the mathematics better. Historians need the mathematician's point of view about what is mathematically important. The mathematician's work determines what it is that most needs a historical explanation. Only the mathematician can tell us which of a half-dozen contemporary concepts is really the crucial one, and which older concepts are worth looking into again -- infinitesimals are one example (See Abraham Robinson's *Non-Standard Analysis* [C29], esp. ch. 10). Furthermore, the mathematician has a better idea of the logical relationship between mathematical ideas, and can suggest connections to the historian which might not be apparent from the historical record alone.

We have seen that the mathematician and the historian bring different skills and different perspectives to their common task of explaining the mathematical present by means of the past. Therefore, as this conference by its existence declares, collaboration between mathematicians and historians can be fruitful. The value of such a collaboration will be enhanced if each collaborator understands the unique contributions which can be made by the other. The importance of the common task, I think, makes it well work the collective efforts.

NOTES

a. By "historian" and "mathematician" I do not mean a classification according to the field of a person's Ph.D., but according to his general point of view. For our present purpose, Dirk Struik and Thomas Hawkins are historians; E. T. Bell and the authors of the *Encyclopädie der mathematischen Wissenschaften* [13] are mathematicians. In general mathematicians and historians have, while writing the history of mathematics, in fact taken the different approaches I describe, though there is no *a priori* reason they would necessarily have to do so.

b. In an approximation to the sum of an infinite series, an 18th century mathematician might take n terms and ask how large the "error" -- the difference between the nth partial sum and the sum of the infinite series -- might be. The series converges in Cauchy's sense when the difference can be made less than any assignable error. Compare his *Cours d'analyse* [B6(2), vol. 3] with his article in the *Comptes Rendus 37* (1853) [B6 (1), vol. 12, pp. 114-124].

c. See Van der Waerden's *Science Awakening* [9, p. 110] and compare Aristotle's *Prior Analytics* i 23, 41a, 26-27.

d. In my article [A5] in the *Am. Math. Mon.* 81, 354-365, I have treated this point at length, especially with respect to changing standards of proof in analysis.

e. The divergence of the harmonic series was proved in the late 17th century by Johann and Jakob Bernoulli, and, for that matter, was shown in the 14th century by Nicole Oresme. For Oresme, see [3, p. 293]; for the Bernoullis, [16, pp. 320-24]

f. See J. d'Alembert, "Réflexions sur les suites et sur les racines imaginaires," *Opuscules Mathématiques*, 1768, vol. 5, pp. 171-83, esp. p. 173. See also Lagrange, *Théorie des Fonctions Analytiques*, 2nd ed., 1813 in [B24, vol. IX, pp. 83-84].

g. For examples of this usage, see d'Alembert, *op. cit.;* L. Euler, "De seriebus divergentibus" in [B11 (1), vol. 15, pp. 586, 588]; and Lagrange, "Sur la résolution des équations numériques," 1772 [B24, II, p. 541].

h. See also d'Alembert, *Dictionnaire raisonné des mathématiques,* which collects the mathematical articles from the *Encyclopédie*, articles "Convergence" and "Serié ou suite." Compare G. S. Kluegel, *Mathematisches Woerterbuch,* 1803, article "Convergirend, Annaehernd."

i. See [B23]; compare [16, pp. 351-68]; and C. Truesdell, "The rational mechanics of flexible or elastic bodies, 1638-1788" in [B11 (2), 11, Sect. 2 (1960)].

j. See [A5 pp. 361-63] and *Leçons sur le calcul des fonctions* (2nd. ed. 1806) in [B24], vol. X, p. 87].

k. See Lagrange, *Traitê de la résolution des équations numêriques de tous les degrês,* 2nd ed., 1808, in [B24, vol. VIII, pp. 46-7, 163]. Compare d'Alembert, *Opuscules mathêmatiques,* 1768, vol. 5, pp. 171-83.

ℓ. J. T. Merz, *History of European Thought in the Nineteenth Century,* vol. I (1904), Dover reprint, 1965.

m. See [B29]; L. Osen, *Women in Mathematics,* M.I.T. Press, 1974; J. L. Coolidge, "Six female mathematicians," *Scripta Math.* 7 (1951), pp. 20-31.

n. A view championed by Lebesgue. See May's biography in H. Lebesgue, *Measure and the Integral* [B27, p. 5].

o. A good introduction to the literature up to 1936, with extensive and humane annotations, is Sarton's *Study of the History of Mathematics* [7]. For a detailed, up-to-date, and extremely valuable guide see May's *Bibliography* [6].

DISCUSSION

The discussion began with three comments by Dieudonné. First, he raised a technical point regarding convergence in the eighteenth century. He claimed that there are many examples to be found among the works of eighteenth-century mathematicians where divergent series are given a sum. Secondly, Dieudonné commented regarding the relationship between mathematics and factors external to its development. He stated that, for example, the general history of the seventeenth century had no connection with Fermat's theory of numbers. Further, in spite of the fact that Descartes, Leibniz, and Cantor were all philosophers and mathematicians, mathematicians in general are *not* philosophers; they *do* mathematics. And thirdly, Dieudonné asserted that simply because the choice of notation may be motivated by circumstances external to mathematical reasoning, does not, however, make this fact significant. He firmly held that such motivating circumstances have no significance.

In her response to Dieudonné's first point, Grabiner emphazized her basic agreement with Dieudonné. Of course it is true, she agreed, that mathematicians in the eighteenth century used divergent series; she had not intended to dispute that fact (See for instance "De seriebus divergentibus" in [B11 (1)vol. 15, pp. 586-588). She re-emphasized her claim that *some* (and not all) historical discussions of infinite series can be illuminated by considering the *two* definitions of convergence used in the eighteenth century.

Kahane emphasized the difference between *definitions* of convergence and *proofs* of convergence. He claimed that already in 1807, Fourier had in mind a definition of convergence based on partial sums, i.e., before Cauchy. [This definition is to be found in Fourier's basic paper on heat conduction submitted to the Academy of Sciences of Paris in 1807. The manuscript is in the library of the Ecole des Ponts et Chaussées and was later published in 1822 as part of his *Théorie analytique de la chaleur* -- *Ed.*]. It would have been possible, stated Kahane, for Fourier to give proof of convergence, but he did not do so. Actually, when Cauchy attempted to prove the convergence of Fourier series, he committed several notable errors, one of which was to mistake absolute convergence for conditional convergence. Ironically, this proved to be helpful, by providing a motivation for Dirichlet to press on, concluded Kahane.

In regard to Dieudonné's second comment, Edwards began by pointing out that precisely in Fermat's time, the rediscovery and translation of the texts of Diophantus occurred. He believed that the events in the general cultural history of the era were extremely relevant to the development of the theory of numbers by Fermat. Hawkins further noted that at least *in the past* many mathematicians were interested in philosophy.

Dou continued in this vein. To him, mathematics is becoming separated from the world, and this is dangerous for mathematics. Even with the Greeks, Dou claimed, mathematics was a real part of life, something to live with. In recent times, there has been a move to separate mathematics and philosophy, and consequently there has arisen a severe need to bridge the gap between the two. Whereas the choice of notation may be irrelevant to mathematics, its philosophy is not.

Putnam concluded this line of thought by noting that the philosophy of mathematics from Plato on has always been of great importance not only for mathematics, but for epistemology and metaphysics as well. Mackey concluded the discussion with a comment on the pedagogical value of the history and philosophy of mathematics to students of mathematics. In Mackey's opinion, both of these approaches to mathematics can be illuminating, but one must question the level of accuracy required. Neither the historian's nor the philosopher's detailed accuracy is beneficial pedagogically. Mackey expressed interest in history, but felt that because of the pressures of his discipline, he could not be interested in too detailed a history.